Combinatorics, 2015 Fall, USTC
Outlines in Weeks 1-2

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Notice: This writing outlines the materials of our lectures, but often does not contain the detailed proofs of statements we proved! All proofs present in lectures may appear in exams.

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In this course, we shall always write $[n] := \{1, 2, 3, ..., n\}$. The size $|X|$ of a finite set X denotes the number of elements in X. Sometime we use the symbol "#" to express the word "number".

Binomial coefficients

- Let X be a set of size n. Define $2^X := \{A : A \subset X\}$ and we show $|2^X| = 2^{|X|} = 2^n$.
- Define $\binom{X}{k} := \{A \subset X : |A| = k\}$.
- **Fact:** $|\binom{X}{k}| = \binom{n}{k} = \frac{n!}{k!(n-k)!}$. That is: the binomial coefficient $\binom{n}{k}$ denotes the number of selections of size k out of n distinct elements.

  In its proof, we also show that # of ordered k-tuple $(x_1, ..., x_k)$ with $x_i \in X$ is equal to $(n)_k := n(n-1)...(n-k+1)$.
- If $n < k$, then $\binom{n}{k} = 0$.
- **Fact:** $\binom{n}{k} = \binom{n}{n-k}$.
- **Fact:** $2^n = \sum_{k=0}^{n} \binom{n}{k}$.
- **Fact:** $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.
- We mention the **Pascal Triangle**, whose number in the nth row and kth column is the binomial coefficient $\binom{n}{k}$. This is related to the previous fact.
- **Fact:** # of integer solutions $(x_1, x_2, ..., x_n)$ to $x_1 + x_2 + ... + x_n = k$ with each $x_i \in \{0, 1\}$ is equal to the binomial coefficient $\binom{n}{k}$.
- **Fact:** # of integer solutions $(x_1, ..., x_n)$ to equation $x_1 + ... + x_n = k$ with each $x_i \geq 0$ = # of labellings of k identical objects using n distinct labels = $\binom{n+k-1}{n-1}$

  We show two proofs of this. The first proof is a combinatorical argument (using k identical "apples" and $n-1$ "walls"). And in the second proof, we define a bijection from the set of solutions to $\binom{[n+k-1]}{n-1}$.

Counting functions

- Define X^Y to be the set of all functions $f : Y \to X$.

- We also can view X^Y as the set of all strings $x_1 x_2 ... x_r$ with elements $x_i \in X$, indexed by elements of Y.
- **Fact:** $|X^Y| = |X|^{|Y|}$.
- **Fact:** The number of **injective** functions $f : [r] \to [n]$ is equal to $(n)_r$.
- **Definition (The Stirling number of the second kind).** Let $S(r, n)$ be the number of partitions of $[r]$ into n unordered non-empty parts.
- Exercise. $S(r, 2) = \frac{1}{2} \sum_{i=1}^{r-1} \binom{r}{i}$.
- **Theorem.** The number of **surjective** functions $f : [r] \to [n]$ is equal to $S(r, n)n!$.
- Any injection $f : X \to X$ is called a **Permutation** of X (also a bijection).

  We may view a permutation in two ways: it is a function from X to X; it also can be think of an arrangement of the elements of X.
- The # of permutations of $[n]$ is $n!$.

<u>The Binomial Theorem</u>

- Define $[x^k]f$ to be the coefficient of term x^k in a polynomial $f(x)$.
- **Fact 1:** Let $f = f_1 f_2 ... f_n$ be a product of polynomials. Then

$$[x^k]f = \sum_{i_1 + ... + i_n = k} \left(\prod_{j=1}^{n} [x^{i_j}] f_j \right).$$

- **Fact 2:** For $j = 1, 2, ..., n$, let

$$f_j(x) := \sum_{i \in I_j} x^i$$

  be a polynomial (note the coefficient of each term is either 1 or 0), where I_j is a set containing nonnegative integers (finite many or infinity). Let $f(x) = f_1 f_2 ... f_n$ be the product of polynomials.

  Then $[x^k]f$ is the number of solutions $(i_1, ..., i_n)$ to $i_1 + ... + i_n = k$ with $i_j \in I_j$ for $j = 1, 2, ..., n$.
- **The Binomial Theorem.** It holds for any real x and any positive integer n that

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

  In the proof we show, we use the Fact 2.

- **Fact.** $\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$.

  We provide two proofs. The first one is a combinatorical proof, that uses a double-counting some combinatorial object. In the second proof, we use the Fact 1 as well as the binomial theorem.

- Exercise. (Vandermonde's Convolution Formula) $\binom{n+m}{k} = \sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j}$.

- **Fact.** $\sum_{\text{all odd } k} \binom{n}{k} = \sum_{\text{all even } k} \binom{n}{k} = 2^{n-1}$.

- Exercise. $n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}$.

  Hint: use the derivatives of the binormial theorem.

Estimating the factorials

- **Theorem.** For any integer $n \geq 1$,

$$e\left(\frac{n}{e}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n.$$

  Here $e = 2.71828...$ is the Euler/natural number. In it proof, we consider the curve of $y = \ln x$ and use rectangles to approach the area bounded below by the curve $y = \ln x$.

- Define $f(n) \sim g(n)$ for functions f and g if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$.

- **Stirling's Formula.** $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

- Exercise. For any integer $n \geq 1$, $n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n$.

  Modify the proof of the upper bound in previous theorem.

Estimating binomial coefficients

- **Fact:** For fixed integer n, view $\binom{n}{k}$ as a function with $k \in \{0, 1, 2, ..., n\}$. It is increasing when $k \leq \lfloor n/2 \rfloor$ and decreasing when $k \geq \lceil n/2 \rceil$.

  In particular, $\binom{n}{k}$ achieves its maximum when $k = \lceil n/2 \rceil$ or $\lfloor n/2 \rfloor$.

- **Fact:** $\frac{2^n}{n+1} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \leq 2^n$.

  It is a corollary of the previous fact.

- Exercise: $\frac{2^n}{\sqrt{2n}} \leq \binom{n}{\frac{n}{2}} \leq \frac{2^n}{\sqrt{n}}$ holds for even n.

  It is a better estimate than the previous one.

- **Fact:** Using Stirling's formula, we have $\binom{n}{\frac{n}{2}} \sim \sqrt{\frac{2}{\pi}}\frac{2^n}{\sqrt{n}}$

- **Fact:** $\binom{n}{k} \leq \frac{n^k}{k!}$

- Exercise. $1 + x \leq e^x$ holds for any real x. (Using calculus)

- **Theorem.** For $1 \le k \le n$,
$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{en}{k}\right)^k.$$
The upper bound can be derived from the facts that $\binom{n}{k} \le \frac{n^k}{k!}$ and $k! \ge e\left(\frac{k}{e}\right)^k$. It also can be viewed as an immediate corollary of the following theorem.

- **Theorem.** For any $1 \le k \le n$,
$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{k} \le \left(\frac{en}{k}\right)^k.$$
This is stronger than the previous upper bound. We prove it using Binomial Theorem (by plugging in a proper value of $x \in [0, 1]$).

Inclusion-Exclusion and its applications

Let $A_1, ..., A_n$ be n subsets of the groud set Ω.

- Definition. Let $A_\emptyset = \Omega$; and for any nonempty subset $I \subseteq [n]$, let
$$A_I = \cap_{i \in I} A_i.$$
For any integer $k \ge 0$, write
$$S_k = \sum_{I \in \binom{[n]}{k}} |A_I|$$
to be the sum of the sizes of all k-fold intersections.

- **Inclusion-Exclusion formula.**
$$|A_1 \cup A_2 \cup ... \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} S_k.$$
Sometime we also use the following version of Inclusion-Exclusion formula,
$$|A_1^c \cap A_2^c \cap ... \cap A_n^c| = |\Omega \setminus (\cup_{i=1}^n A_i)| = \sum_{k=0}^{n} (-1)^k S_k,$$
where $A_i^c = \Omega \setminus A_i$ means the complement of subset A_i. We point out that $S_0 = |A_\emptyset| = |\Omega|$. It also holds that
$$|A_1^c \cap A_2^c \cap ... \cap A_n^c| = \sum_{I \subset [n]} (-1)^{|I|} |A_I|.$$

- We show two proofs. In the first proof of this formula, we write $A := A_1 \cup A_2 \cup ... \cup A_n$; for any subset $X \subseteq \Omega$, we define its characterization function $\mathbf{1}_X : \Omega \to \{0, 1\}$ by assigning $\mathbf{1}_X(x) = 1$ if $x \in X$ and $\mathbf{1}_X(x) = 0$ if $x \in \Omega \backslash X$. So we have $\sum_{x \in \Omega} \mathbf{1}_X(x) = |X|$. The key ingredient in this proof is that

$$(\mathbf{1}_A - \mathbf{1}_{A_1})(\mathbf{1}_A - \mathbf{1}_{A_2})...(\mathbf{1}_A - \mathbf{1}_{A_n}) = 0$$

holds for all $x \in \Omega$. Then we expand the above product into a summation of 2^n terms. By summing over all $x \in \Omega$, the above expression of 2^n terms becomes the Inclusion-Exclusion formula.

The second proof considers the contributions of each element $a \in \Omega$ to both sides. And we show: for each $a \in \Omega$, the contributions of a to both sides always equal.

- Definition. Let $\varphi(n)$ be the number of integers $m \in [n]$ which are relatively prime to n. Here, m is relatively prime to n means that the greatest common divisor of m and n is 1.

- **Fact:** If n can be written as $n = p_1^{a_1} p_2^{a_2} ... p_t^{a_t}$, where $p_1, ..., p_t$ are distinct primes in $[n]$, then

$$\varphi(n) = n \prod_{i=1}^{t} \left(1 - \frac{1}{p_i}\right).$$

We proved this by considering $\Omega = [n]$ and the sets $A_i = \{m \in [n] : p_i | m\}$ for $i = 1, ..., t$. Note that $\varphi(n) = |\Omega \backslash (\cup_{i=1}^t A_i)|$

- Definition. A permutation $\sigma : X \to X$ is called a **derangement** of X if $\sigma(i) \neq i$ for any $i \in X$. We use D_n to denote the set of all derangements of $[n]$.

- **Fact:** For any integer $n \geq 1$,

$$|D_n| = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

We apply inclusion-exclusion by considering $A_i = \{\sigma | \sigma(i) = i\}$ for $i = 1, ..., n$.

- **Fact:** $|D_n| \sim \frac{n!}{e}$.

It is because $\lim_{n\to\infty} \frac{|D_n|}{(n!/e)} = e \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e \cdot e^{-1} = 1$, by Taylor's series $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

- Recall. (i) $S(n, k)$ is the number of partitions of a set of size n into k nonempty parts.

(ii) $S(n, k)k!$ is the number of surjective functions from Y to X, where $|Y| = n$ and $|X| = k$.

- **Fact:**

$$S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k - i)^n.$$

To prove this, we use inclusion-exclusion (again!) by considering $\Omega = X^Y$ and its subsets $A_i := \{f : Y \to X \backslash \{i\}\}$.

- Exercise. If p is odd integer, then

$$|A_1 \cup A_2 \cup ... \cup A_n| \le \sum_{k=1}^{p} (-1)^{k-1} S_k;$$

if p is even, then

$$|A_1 \cup A_2 \cup ... \cup A_n| \ge \sum_{k=1}^{p} (-1)^{k-1} S_k.$$

Generating functions

- **Definition.** The (ordinary) generating function (or GF for short) for an infinity sequence $a_0, a_1, ...$ is a power series

$$f(x) = \sum_{n \ge 0} a_n x^n.$$

We have two ways to view the generating function.

(i). When the power series $\sum_{n\ge 0} a_n x^n$ converges (i.e., there exists a radius $R > 0$ of convergence), we view G.F. as a function of x and we can apply operations of calculus on it, including differentiation and integration. For example, in this case we know that

$$a_n = \frac{f^{(n)}(0)}{n!}.$$

Also recall the following sufficient condition on the radius of convergence that if $|a_n| \le K^n$ for some constant $K > 0$, then $\sum_{n\ge 0} a_n x^n$ converges in the interval $(-\frac{1}{K}, \frac{1}{K})$.

(ii). When we are not sure of the convergence, we view G.F. as a formal object with additions and multiplications. Let $a(x) = \sum_{n\ge 0} a_n x^n$ and $b(x) = \sum_{n\ge 0} b_n x^n$.

Addition.

$$a(x) + b(x) = \sum_{n\ge 0} (a_n + b_n) x^n.$$

Multiplication. Let $a(x)b(x) = \sum_{n\ge 0} c_n x^n$, where

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}.$$

- $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ holds for any real x with $|x| < 1$. By the point view of (i), we can compute the derivatives of two sides to get more identities, i.e. the first derivative will give

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1}.$$

- Exercise. For all integers $k \ge 0$, let $a_{2k} = 0$ and $a_{2k+1} = 1$. Find the GF of this sequence.

- Recall the following facts:

  Fact 1. If $f(x) = \prod_{i=1}^{k} f_i(x)$ for polynomials $f_1, ..., f_k$, then

$$[x^n]f = \sum_{i_1+i_2+...+i_k=n} \prod_{j=1}^{k} \left([x^{i_j}]f_j\right).$$

  Fact 2. For $j = 1, 2, ..., n$, let

$$f_j(x) := \sum_{i \in I_j} x^i$$

  where I_j is a set containing nonnegative integers. Let $f(x) = f_1 f_2 ... f_n$ be the product.

  Then $[x^k]f$ is the number of solutions $(i_1, ..., i_n)$ to $i_1 + ... + i_n = k$ with $i_j \in I_j$ for $j = 1, 2, ..., n$.

- **An equivalent form** of the Fact 1, which will be very useful in the use of generating functions.

  For $j = 1, 2, ..., n$, let $f_j(x) = \sum_{i \in I_j} x^i$. Define b_k to be the number of solutions to $i_1 + i_2 + ... + i_n = k$ with each $i_j \in I_j$. Then

$$\prod_{j=1}^{k} f_j(x) = \sum_{k=0}^{\infty} b_k x^k.$$

- Problem. Let $a_0 = 1$ and $a_n = 2a_{n-1}$ for $n \geq 1$. Find a_n.

  We let $f(x) = \sum a_n x^n$ be the generating function. Then we show $f(x) = 1 + 2xf(x)$,so $f(x) = \frac{1}{1-2x}$, which implies that $f(x) = \sum 2^n x^n$ and therfore $a_n = 2^n$.

- From the above problem, we see one of the basic ideas for using GF: in order to find the general expression of a_n, we work on its GF $f(x)$; once we find the formula of $f(x)$, then we can expand $f(x)$ into a power series and find a_n by choosing the coefficient of the right term.

Recurrence relation and the Newton's binomial theorem

- Problem. Let A_n be the set of strings of length n with entries from the set $\{a, b, c\}$ and with NO "aa" occuring (in the consecutive positions). Find $a_n = |A_n|$ for $n \geq 1$.

- We first observe that $a_1 = 3, a_2 = 8$ and for any $n \geq 2$

$$a_n = 2a_{n-1} + 2a_{n-2},$$

  therefore $a_0 = 1$. Let $f(x) = \sum_{n \geq 0} a_n x^n$. Then we use the recurrence relation to get

$$f(x) = 1 + 3x + 2x(f(x) - 1) + 2x^2 f(x),$$

which implies that

$$f(x) = \frac{1+x}{1-2x-2x^2}.$$

By Partial Fraction Decomposition, we calculate that

$$f(x) = \frac{1-\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}+1+2x} + \frac{1+\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}-1-2x},$$

which implies that

$$a_n = \frac{1-\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}+1} \left(\frac{-2}{\sqrt{3}+1}\right)^n + \frac{1+\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}-1} \left(\frac{2}{\sqrt{3}-1}\right)^n.$$

Note that a_n must be an integer but its expression is of a combination of irrational terms! Observe that $\left|\frac{-2}{\sqrt{3}+1}\right| < 1$, so $\left(\frac{-2}{\sqrt{3}+1}\right)^n \to 0$ as $n \to \infty$. Thus, when n is sufficiently large, a_n is about the value of the second term $\frac{1+\sqrt{3}}{2\sqrt{3}} \frac{1}{\sqrt{3}-1} \left(\frac{2}{\sqrt{3}-1}\right)^n$; equivalently a_n will be the nearest integer to that.

- Definition. For any real r and integer $k \geq 0$, let

$$\binom{r}{k} = \frac{r(r-1)...(r-k+1)}{k!}.$$

- **Newton's Binomial Theorem.** For any real r,

$$(1+x)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^k$$

holds for any $x \in (-1, 1)$.

The proof is using Taylor series which we did not cover. Note that the Binomial Theorem says that for positive integer n, $(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$ holds for any real x!

- **Corollary.** Let $r = -n$ for integer $n \geq 0$. Then $\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$. Therefore

$$(1+x)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k.$$